The Beauty of Numbers

Johann Sebastian Bach, Le Corbusier and Maurits Escher: Mathematics has influenced many a creative genius. But also mathematics itself contains an element of beauty. Our author, in any case, is firmly convinced of this, and is greatly inspired by its conciseness, simplicity, clarity and the absolute persuasiveness of its arguments and ideas: a testimony to how much pleasure the art of mathematics can bring – a pleasure that can reveal itself to just about anyone.

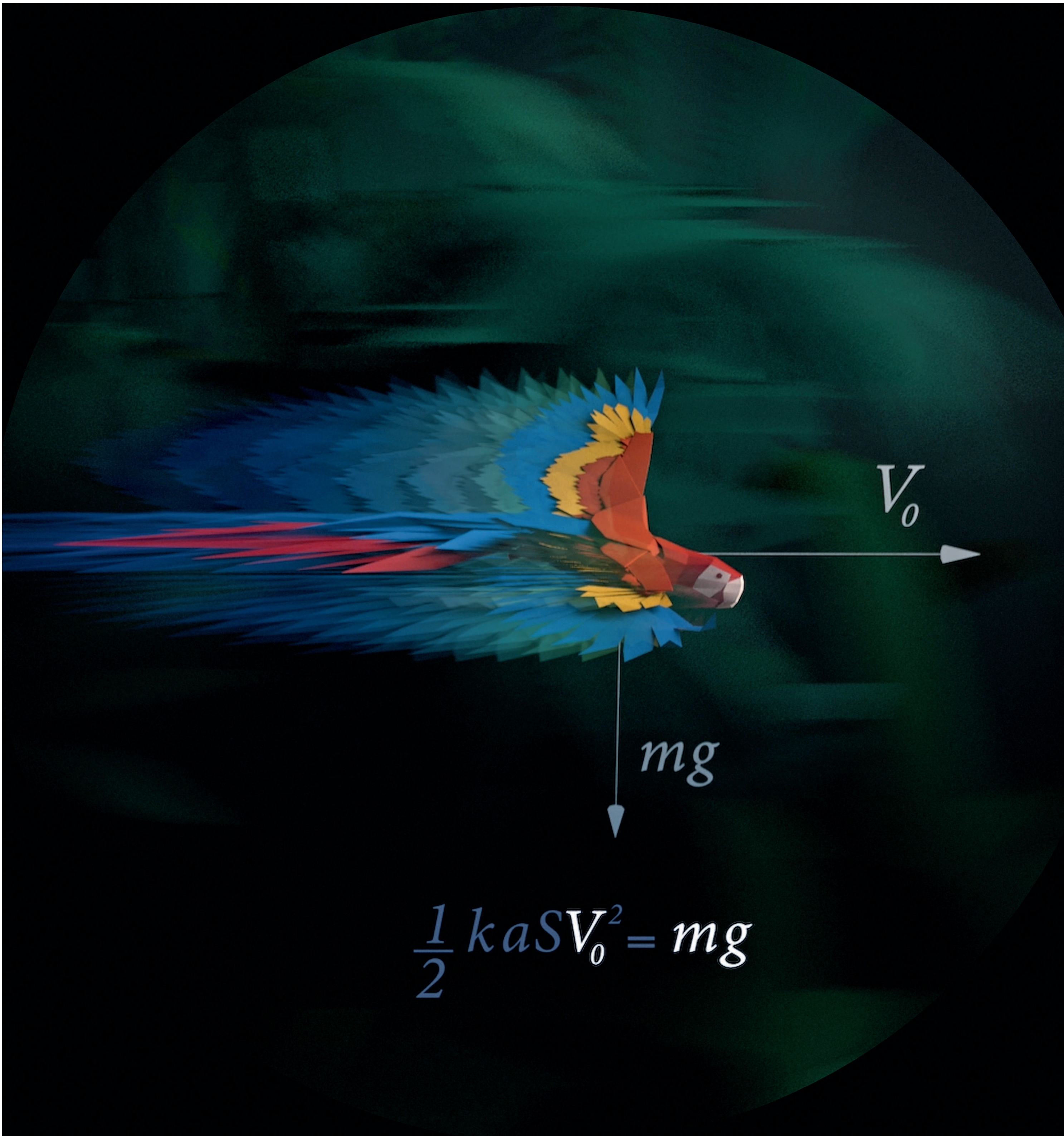
TEXT **DON ZAGIER**

A Hungarian number theorist once defined mathematicians thus: "A mathematician is a machine for transforming coffee into theorems." At my institute in Bonn, good coffee is scarce, but there is no lack of mathematicians or theorems, leading me to wonder occasionally whether it wouldn't be possible to use mathematicians in reverse!

What does a mathematician do? Invent or discover new things?

Now, there are some people who can't stop processing coffee like this, and others who are filled with dread at the mere thought of mathematics. I will come back to this later. But first I want to address another issue: What is mathematics? What makes it beautiful? And what can be done to convey the joy of mathematics to non-mathematicians as well?

The question as to what mathematics actually is sounds naive. In fact, however, it is not so easy to answer, and indeed, philosophers have been pondering this issue for centuries. At the very beginning of his *Critique of Pure Reason,* Immanuel Kant even asked how pure mathematics was possible. Other sciences are clearly characterized according to the objects they study: heavenly bodies, living things, human relationships or whatever. With mathematics, it is not so easy. For one thing, mathematics doesn't always deal with the same objects. Numbers, algebraic formulas, analytical functions and geometric shapes are obviously included, but mathematical thinking is actually more about studying structures per se than about studying the structure of specific, predetermined ob-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jects. And the problem goes even deeper: unlike other disciplines, it isn't even clear where the objects we study are actually found. Are they internal or external? Subjective or objective? Do they exist only in our minds or in the real world? Is a mathematician's task to invent or to discover?

The fact that mathematical results can be verified objectively speaks in favor of discovering: The proof a mathematician provides for a theorem will convince any mathematician about the truth of the assertion, as long as he hasn't made a mistake. Different mathematicians investigating the same problem will always come up with the same answer, regardless of their personality or individual preferences. The same also applies to different cultures that have often come across precisely the same mathematics quite independently of each other. For instance, the formula for solving quadratic equations, the Pythagorean theorem (which is not called this everywhere, of course!) and the algorithm for taking cube roots were discovered in many different cultures in ancient times.

> There are a number of valid consequences at any time

But mathematicians are just as often inventors. One indication of this is that they often have the purely subjective feeling of creating something of their own. Also, different mathematicians are often led to different problems, and therefore to different results, depending on their personal preferences and experience, and they can often even be recognized from their mathematical theorems – like a personal trademark. In precisely the same way, different cultures have often taken completely different mathematical directions and thus developed their own mathematics. The Greeks, for instance, invented the concept of proof and focused on this, while the Chinese often made the same discoveries but presented them as algorithms or arithmetical recipes. Or to name another example: The Egyptians used mathematics in commerce, surveying and astronomy, as did other ancient peoples, and developed arithmetical methods that used rational numbers (fractions), but in a very unusual way. Instead of writing the fractions as quotients of numerators and denominators, they allowed only purely reciprocal numbers (1/n), and represented every fraction as the sum of such numbers. Moreover, they allowed only different denominators: 2/5 was not written as 1/5 plus 1/5, but as 1/3 plus 1/15!

So where does the truth lie? Most mathematicians would say the truth is a combination of both aspects. At any time and for any problem there are a number of valid consequences that follow from the axioms and what is already known, similar to the many moves that are possible for any position in a game of chess. These consequences are already there in a certain sense, but the mathematician has to decide which path to take each time – and this is where their individual skills, preferences and personality come into play. The French mathematician Gustave Choquet put it thus: "The theorem that one is seeking has existed from time immemorial. But in order to discover it, one must invent a path."

Is mathematics an art or a science? Again, both points of view can easily be defended. In favor of it being an art is the fact that mathematics can often be found in art (in the conventional sense): in architecture, one need only think of the pyramids, the Parthenon or the buildings of Christopher Wren, Le Corbusier or any number of other architects; in music, the works of Bach, Mozart or Schönberg; and in painting, Dürer or da Vinci.

Moreover, mathematics itself can be aesthetically beautiful – such as certain geometric shapes, like the five regular polyhedra discovered by Plato or, to use a more modern example, the beautiful fractal images that are familiar to many. Occasionally, art even leads to new mathematics, as is the case with many

Image from the film *Les Paradis mathématiques*, 2011. Création BUF

of the drawings by Dutch artist Maurits Escher, for instance. An even more interesting example is that of the so-called tessellations of the plane. A difficult mathematical theorem states that there are exactly 17 fundamentally different types of symmetry that such a tessellation can have; surprisingly, all 17 were discovered by Islamic artists in the Middle Ages and incorporated into wonderful ornamentations in the Alhambra in Granada.

But when I speak of art and mathematics, I do not mean these relationships between mathematics and the other arts, as diverse and interesting as they may be, but rather that mathematics itself is art. The relevant aesthetic criteria here are not so much visual in nature as abstract: the conciseness, simplicity, clarity and absolute persuasiveness of the arguments and ideas. At first sight, these criteria may seem intellectual rather than artistic, but hardly anyone who has worked with mathematics for a lengthy time fails to develop a sense of its beauty. Mathematicians use words like “beautiful” and “elegant” even more frequently than scientific terms like “convincing” or “correct”. Even more interesting is the fact that this feeling for mathematical beauty very often turns out to be the surest guide when choosing the best way through the labyrinth of mathematics – a kind of Ariadne’s thread.

Artists can apply aesthetic criteria when making a decision: What should I write? What should I paint? What should I compose? Scientists only rarely have this luxury, since nature has not always chosen the path that pleases us humans best. Mathematics is somewhere in between: mathematicians don‘t necessarily have to (and certainly can‘t always) proceed according to aesthetic criteria, but in the vast majority of cases, the mathematically best way turns out to also be the best one from an aesthetic point of view. There is no better strategy than to always look for the most beautiful solution.

Mathematics can therefore quite easily be considered to be an art. But there are also good arguments for classifying it as a science. Mathematics has a degree of objectivity that the other sciences scarcely attain: its results are absolutely certain, because they are proven. And once something has been discovered, it never goes out of date. Subsequent developments may add new aspects, but a truth, once revealed, never changes.

Mathematics can even be considered to be more scientific than the other sciences, because it is less dependent on the accidental properties of our world. The various sciences could thus be ordered from soft science to hard science, for example history – sociology – psychology – medicine – biology – chemistry – physics and, only then, mathematics. The history of a country is largely determined by chance events and could quite easily have turned out differently; the sociology of a people greatly depends on cultural aspects; psychology is more universal, but still culture dependent; medicine applies to all cultures and peoples, but only to the species humankind; biology applies to all known forms of life, but would presumably look different on a distant planet; chemistry would be unchanged even on distant planets, but under conditions of extreme temperature or extreme pressure – after the Big Bang, for example – would obey other laws. And even physics, which appears to be universal, is not necessarily so, since one can easily imagine a different universe in which, say, the ratio of the mass of a proton and an electron has a value other than 1,836. But mathematics would be valid even in this other universe: two plus two would still equal four, and every number would still be a product of prime numbers. It is a paradox: mathematics, the seemingly most unreal of all the sciences, describes the most real reality! >

Nature has not always chosen the path that pleases us humans best

Why, then, does mathematics give so many people so much pleasure? The obvious answer, and one that is certainly not completely wrong, is that solving problems and doing difficult puzzles is, quite simply, fun. To this is added the aesthetic feeling described above, the pleasure obtained from the elegance and beauty of the results and arguments that one has read in the work of others or has discovered for oneself. But I think that the main source of the satisfaction felt by mathematics adepts derives from that special feeling of discovering, without external aid, a part of the truth, of unearthing one of nature's secrets.

We manage to prove an answer that is irrefutable for all eternity

As a simple example of this, I would like to present Euclid's famous proof for the existence of infinitely many prime numbers:

Imagine that there were only a finite number of prime numbers, say 2, 3, 5, 7 and so on, up to 31, and then no others. One could multiply all of these prime numbers 2, 3, ... , 31 together and add 1 to the product. The resulting number would not be divisible by any of the prime numbers 2, 3, . . . , 31, because it is greater by one than a multiple of each of them. Like any number, however, it would either have to be a prime number itself or contain a prime factor smaller than itself. Contrary to the hypothesis, this factor would be a prime number that was not included in our original list.

Whether or not all the details of this argument can be understood after such a brief exposition, I believe that everyone can certainly recognize one truly fantastic property of this argument: We start off with a question (is there a finite or an infinite number of prime numbers?) that we humans actually should not be able to answer at all, because of course we can never consider more than a small, finite fraction of the prime numbers. Nevertheless, we manage to find the answer by using a couple of simple yet very subtle sentences to prove it irrefutably for all eternity. Mathematics, which comes from the inside and yet describes something on the outside, is the only science where thought alone can be used, not only to find the truth, but to prove it as well. And it is a fantastic feeling to be able to do this.

Mathematics, then, can give some people a deep feeling of joy. But unfortunately, only a few: it is definitely not for everyone. In contrast to music and good food, about which some people are passionate and some not all that interested, but which almost everybody appreciates to some extent, mathematics prompts very different feelings: those who have discovered its fascination are hooked forever, while most people cannot begin to imagine how mathematics and pleasure can be related at all. I don't want to go into the reasons for this, even though there are some very interesting studies on the subject. (What is certain is that culture plays a major role.) But actually I am convinced that many people have the potential to love mathematics.

The main problem may be that most people have never seen real mathematics: The mathematics that everyone learns in school is almost always just a collection of recipes for everyday use or, at best, in science. Beautiful mathematics is hard to find. But in order to understand the beauty of mathematics, one must have also encountered it. Imagine that you knew that music existed, but that you had never heard a single sound or melody. It would be just as difficult for you to convince yourself of its beauty as it is for many people to convince themselves that mathematics is beautiful. Of course, it would be even better if you had not only heard a few sounds and melodies, but had also played or sung them yourself. And even better if you had done this as a child! It is just the same with mathematics.

Fortunately, encounters with mathematics are very much possible. There are many mathematical results whose formulations (and occasionally proofs,

Image from the film *Les Paradis mathématiques*, 2011. Création BUF as well) can be understood by non-mathematicians, and whose beauty can surely be understood by many. Examples include the above-mentioned Platonic solids, Euler's formula and Lagrange's theorem, which states that every natural number is the sum of at most four square numbers. There are also others with which one can experiment oneself, and thus experience the pleasure of mathematical discovery.

I can clearly remember, as a 12-year-old, being told about Pick's theorem by a chemist. This theorem states that the area of a polygon drawn on graph paper, with its corners lying only on lattice points, is one less than the number of interior points plus half the number of points on the perimeter. I pondered this for weeks before I finally found a proof. Or the mysterious Möbius strip with only one surface and one edge: try to find out, by pure thought, what happens if you cut such a strip down the middle or a third of the way from the edge.

I pondered about Pick's theorem for weeks

These types of encounters, which both young and old might find absolutely fascinating, can be taught by a good teacher or a good book. Two books by Hans Magnus Enzensberger come to mind: *The Number Devil* and *Drawbridge Up,* the former more for children, and the latter more for adults. But in addition to teachers and textbooks, there is a third way: a museum for mathematics where one can see, hear and touch beautiful things. One such example was the exhibition at the Fondation Cartier in Paris: "Mathematics: A Beautiful Elsewhere," where eight artists helped visitors experience the aesthetics of mathematics. Also the catalog of this unique experiment invites readers to take a journey into the beauty of abstract thought. ◂

Photo: John Petter Reinertsen

THE AUTHOR

Don Zagier, born in 1951, has been a Director at the Max Planck Institute for Mathematics in Bonn since 1995 and a professor at the Collège de France in Paris since 2000. This essay by the internationally renowned number theorist is based on a speech he gave in 2003 on the occasion of the opening of math.space in Vienna's Museum Quarter, and on the English/French version published in the catalog to the exhibition "Mathematics: A Beautiful Elsewhere."

Further information about this interactive show, in which Don Zagier played a crucial role, can be found at www.fondation.cartier.com